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On similarity solutions occurring in the theory of interactive laminar boundary layers

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A theoretical investigation of similarity solutions for interactive laminar boundary layers is presented. The questions of uniqueness and of the appearance of homogeneous eigensolutions are discussed. The similarity solutions yielding the asymptotic behaviour of the nonlinear triple-deck equations in the far field can be used either to improve the development of computational schemes or to check the accuracy of numerical results. A special similarity solution governed by a modified Falkner–Skan boundary-value problem determines the shape of a wall generating the largest possible deflection of a laminar boundary layer in supersonic flow if separation is to be avoided. Increasing the controlling parameter of this special pressure distribution (for both supersonic and subsonic flows) beyond a cutoff value leads to a global breakdown of the interacting laminar-boundary-layer approach which cannot be removed or avoided.

1. Introduction

This paper presents and discusses similarity solutions of the nonlinear triple-deck problem for supersonic and subsonic two-dimensional flows. The origins, theory and applications of triple-deck structures have been extensively reviewed (Stewartson 1974, 1981; Smith 1982; Messiter 1983; Kluwick 1987) and so it seems sufficient to comment that perhaps the most significant aspect of triple-deck theory is that it governs the onset of external separation as well as describing some fundamental attached-flow problems. A number of recent studies deal with the still not fully known properties that arise for larger disturbances, particularly those yielding relatively large-scale separations and eddies. Of special interest in this context is the question whether a singularity terminates the triple-deck structure for increasing disturbances of the flow, cf. Smith & Khorrami (1991). Another class of recent investigations concerns the unsteady effects of surface-mounted obstacles on the boundary-layer flow over a surface, including instability and the transition to turbulence, cf. Bodonyi *et al.* (1989).

In all these problems it is necessary to have a very accurate solution of the steady basic flow. For the development of a computational scheme it is helpful to know exactly the asymptotic properties of the field quantities far upstream and downstream. This asymptotic behaviour can directly be incorporated in the numerical formulation of the boundary conditions and/or it can be used as an accuracy check of the computational results.

In this paper we study similarity solutions which describe the asymptotic behaviour of triple-deck flows along general power-law-profile contours. The question of existence and uniqueness of these solutions is investigated in connection with the possibility of the appearance of homogeneous eigensolutions. After discussion of some applications of the results we finally present a special similarity solution which is governed by a nonlinear boundary-value problem of the Falkner–Skan type. As a special result we obtain the shape of a wall generating the largest possible deflection of a laminar boundary layer in supersonic flow or – corresponding to subsonic flow – the pressure distribution producing the sharpest pressure rise in the shortest distance without separation.

2. Problem formulation

It will be assumed that the flow structure has the triple-deck form introduced by Stewartson (1969), Neiland (1969) and Messiter (1970) and that the interaction process is caused by a distortion of the surface $y = F(x)$ and/or by an incoming wave in supersonic flow). After application of Prandtl's transposition theorem ($y \to y + F(x)$, $v \to v + u\,\mathrm{d}F/\mathrm{d}x$) the governing lower-deck equations expressed in terms of scaled variables are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial^2 u}{\partial y^2}, \tag{1b}$$

with

$$u = v = 0 \quad \text{at} \quad y = 0, \tag{1c}$$

$$u \to y \quad \text{as} \quad x \to -\infty, \tag{1d}$$

$$u \to y + A(x) \quad \text{as} \quad y \to \infty. \tag{1e}$$

Here (x, y) are the streamwise and transverse Cartesian coordinates, respectively, (u, v) are the corresponding velocity components, and $-A(x)$ is the displacement increment of the boundary layer due to the interaction process. In supersonic flow the induced pressure $p(x)$, the displacement $-A(x)$ and the body shape $F(x)$ satisfy Ackeret's relationship

$$p(x) = -\frac{\mathrm{d}A}{\mathrm{d}x} + \frac{\mathrm{d}F}{\mathrm{d}x} \quad (M_\infty > 1), \tag{1f}$$

whereas in subsonic flow these quantities are related by the Cauchy–Hilbert integral

$$p(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{A'(\xi) - F'(\xi)}{x - \xi}\,\mathrm{d}\xi \quad (M_\infty < 1). \tag{1g}$$

Let us assume that the pressure far from the interaction region is given by the power-law distribution

$$p(x \to \infty) \sim \alpha D x^\beta, \tag{2a}$$

$$p(x \to -\infty) \sim \alpha \bar{D}(-x)^\beta. \tag{2b}$$

Here α denotes a measure for the distortion of the surface to be given in (3). The constants D and $\bar{D}$ which are different in the case of sub- or supersonic flow will be presented in (4). A simple example of a body contour producing such a pressure distribution consists of a flat plate upstream of the interaction region and of a power-law profile downstream of it:

$$F(x \to \infty) \sim \alpha x^{\beta+1}, \quad F(x \to -\infty) \equiv 0. \tag{3}$$

In the case of supersonic flow (1 f) predicts a non-zero pressure distribution for $x \to \infty$ only:

$$D = \beta + 1, \quad \bar{D} = 0 \quad (M_\infty > 1). \tag{4a}$$

Evaluation of (1 g) for the same profile yields the coefficients of the pressure distribution (2) in subsonic flow:

$$D = -\frac{\beta+1}{\tan[\pi(\beta+1)]}, \quad \bar{D} = \frac{\beta+1}{\sin[\pi(\beta+1)]} \quad (M_\infty < 1). \tag{4b}$$

This result, obtainable by means of complex variables or Fourier transform theory, is applicable only if the exponent β is non-integer. An additional constraint $\beta > -2$ is necessary also, since a hump or an indentation or an equivalent distortion of the displacement thickness caused by the nonlinear interaction process in the region $x = O(1)$ produces a contribution of $O(|x|^{-2})$ to the far field of the pressure according to (1 g).

Then the question arises of whether it is possible to specify unique similarity solutions to the lower-deck equations. As a first step let us discuss the behaviour of the flow quantities far downstream where the pressure is given by (2 a). The appropriate similarity form for the stream function Ψ is found to be

$$\Psi(x \to \infty, y) \sim \tfrac{1}{2}y^2 + \alpha x^\beta f(\eta) + O(\alpha x^{\beta - \frac{4}{3}}) + O(\alpha^2 x^{2\beta - \frac{2}{3}}) \tag{5a}$$

with the similarity variable

$$\eta = y / x^{\frac{1}{3}}. \tag{5b}$$

For the velocity components and the shear stress we obtain (a prime denotes $\mathrm{d}/\mathrm{d}\eta$)

$$u(x \to \infty, y) \sim y + \alpha x^{\beta - \frac{1}{3}} f'(\eta), \tag{5c}$$

$$v(x \to \infty, y) \sim -\alpha x^{\beta-1}[\beta f(\eta) - \tfrac{1}{3}\eta f'(\eta)], \tag{5d}$$

and

$$\tau(x \to \infty, y) = \frac{\partial u}{\partial y} \sim 1 + \alpha x^{\beta - \frac{2}{3}} f''(\eta). \tag{5e}$$

Substitution of (5 c) into (1 e) yields

$$A(x \to \infty) \sim \alpha K_1 x^{\beta - \frac{1}{3}} \quad \text{with} \quad K_1 = f'(\eta \to \infty). \tag{6}$$

This displacement thickness causes a pressure response of $O(\alpha x^{\beta - \frac{4}{3}})$ which gives rise to the corresponding term in (5 a). The details of a general continuation of the expansion (5 a) are omitted here, but the last term, $O(\alpha^2 x^{2\beta - \frac{2}{3}})$ – evoked by the nonlinearity of the momentum equation (1 b) – shows that the leading term $\alpha x^\beta f(\eta)$ gives the correct asymptotic description for $x \to \infty$ if $\beta < \frac{2}{3}$. Therefore $\beta = \frac{2}{3}$ represents an upper limit where the validity of (5), (6) is restricted to small disturbances $\alpha \ll 1$. No similarity solution can be found for pressure distributions (2) with $\beta > \frac{2}{3}$ and even a solution to the triple-deck equations seems not to exist in this case, as will be shown later (see §5).

Substituting (5) into the equation of motion (1 b) and retaining the highest-order terms only leads to the linear ordinary differential equation

$$f''' + \tfrac{1}{3}\eta^2 f'' - \beta(\eta f' - f) - \beta D = 0, \tag{7}$$

where the constant βD characterizes the magnitude of the pressure gradient

$$\frac{\mathrm{d}p}{\mathrm{d}x} \sim \alpha \beta D x^{\beta - 1} \quad \text{for} \quad x \to \infty, \tag{8}$$

following from (2) and (4). The no-slip condition at the wall (1c) together with (5c) and (5d) requires

$$f(0) = f'(0) = 0. \qquad (9a)$$

The demand for a finite displacement thickness leads to the third boundary condition

$$f'(\eta \to \infty) \sim K_1 + o(1), \qquad (9b)$$

already formulated in (6).

An asymptotic expansion of the differential equation (7) in the limit $\eta \to \infty$,

$$f(\eta \to \infty) \sim D + K_1 \eta + K_2 \eta^{3\beta} + K_3 \eta^{-3\beta-4} \mathrm{e}^{-\eta^3/9} + \ldots \quad \text{for} \quad \beta \neq \tfrac{1}{3}, \qquad (10a)$$

$$f(\eta \to \infty) \sim D + K_1 \eta + K_2 \eta \ln \eta + K_3 \eta^{-5} \mathrm{e}^{-\eta^3/9} + \ldots \quad \text{for} \quad \beta = \tfrac{1}{3}, \qquad (10b)$$

shows that the boundary condition (9b) is always satisfied for $\beta < \frac{1}{3}$ and therefore a unique solution of the boundary-value problem (7), (9) can only be specified in the case $\beta \geqslant \frac{1}{3}$. One might argue that only on profile contours (3) with $\beta \geqslant \frac{1}{3}$ is the downstream acting pressure gradient (8) strong enough to determine the similarity solution (5) in a definite way. As shown in the next section, however, it will be possible to decrease this lower bound for the exponent β by formulating a stronger outer constraint for $f(\eta \to \infty)$ replacing condition (9b).

For completeness it should be noted that more general pressure distributions than (2) including logarithmic terms $p \sim \alpha D x^{\beta} (\ln x)^{\delta}$ in general lead to modified expressions (5) of the form $\psi \sim \frac{1}{2} y^2 + \alpha x^{\beta} (\ln x)^{\delta} f(\eta)$. One important exception is the ramp geometry ((3) with $\beta = 0$) in subsonic flow where the pressure $p \sim -(\alpha/\pi) \ln |x|$, as $|x| \to \infty$, leads to the similarity structure $\psi \sim \frac{1}{2} y^2 + \alpha f(\eta)$ containing no log terms. In this case the governing equation $f''' + \frac{1}{3} f'' \eta^2 + 1/\pi = 0$ has the asymptotic expansion $f \sim (3/\pi) \ln \eta + K_1 \eta + K_2 + K_3 \eta^{-4} \mathrm{e}^{-\eta^3/9}$ also differing from (10).

3. The determination of unique similarity solutions

Let us now turn to the discussion of the flow properties far upstream of the interaction region. If a forcing pressure term is present for $x \to -\infty$, (2b), the stream function can be expressed in a form analogous to (5):

$$\Psi(x \to -\infty, y) \sim \tfrac{1}{2} y^2 + \alpha (-x)^{\beta} \bar{f}(\bar{\eta}) + \ldots \quad \text{with} \quad \bar{\eta} = y/(-x)^{\frac{1}{3}}. \qquad (11)$$

Of course, such a non-vanishing pressure distribution exists in subsonic flows but also in supersonic motions along non-flat body profiles with $F(x \to -\infty) \neq 0$. The function $\bar{f}(\bar{\eta})$ satisfies the differential equation

$$\bar{f}''' - \tfrac{1}{3} \bar{\eta}^2 \bar{f}'' + \beta (\bar{\eta} \bar{f}' - \bar{f}) + \beta \bar{D} = 0 \qquad (12)$$

and, for large $\bar{\eta}$, is thus given by

$$\bar{f}(\bar{\eta} \to \infty) \sim \bar{D} + \bar{K}_1 \bar{\eta} + \bar{K}_2 \bar{\eta}^{3\beta} + \bar{K}_3 \bar{\eta}^{-3\beta-4} \mathrm{e}^{-\eta^3/9} + \ldots, \qquad (13)$$

except for $\beta = \frac{1}{3}$, where the same $\bar{K}_2 \bar{\eta} \ln \bar{\eta}$ term appears as in (10b). Expansion (13) contains an exponentially growing term as $\bar{\eta} \to \infty$ which is inadmissible. Hence the boundary-value problem for $\bar{f}(\bar{\eta})$ – consisting of (12), the wall boundary conditions (9a) and the requirement for the disappearance of the exponentially increasing complementary function as a third boundary condition in the limit $\bar{\eta} \to \infty$ – always yields unique solutions, independent of the value of the exponent β. These unique similarity solutions for $x \to -\infty$ are well known in the triple-deck literature (cf. Brown & Stewartson 1970, eq. (5.3); Smith & Merkin 1982, eq. (2.2a, b)).

If (13) and (10) – valid far upstream and downstream – are inserted into the similarity expansions (11) and (5), respectively, it is seen that the terms $\bar{K}_2\bar{\eta}^{3\beta}$ and $K_2\eta^{3\beta}$ produce a contribution of $O(y^{3\beta})$ to the stream function $\Psi(x, y\rightarrow\infty)$. This indicates that the two disturbances (for $x\rightarrow-\infty$ and $x\rightarrow\infty$) can be connected by an expansion of the stream function at the outer edge of the lower deck: $y\rightarrow\infty$. By insertion of terms of the form $y^r(\ln y)^s F_{rs}(x)$ into the nonlinear momentum equation (1*b*) and the relation (1*e*) and retaining the highest-order terms in y only we get

$$\Psi(x, y\rightarrow\infty) \sim \tfrac{1}{2}[y+A(x)]^2+p(x)+K_{rs}\,y^r(\ln y)^s+\dots \quad \text{with} \quad r<2. \tag{14}$$

This expansion, which is valid for all x, contains the pressure $p(x)$, the displacement function $A(x)$, free constants K_{rs} but no arbitrary functions $F_{rs}(x)$. If (14) is carried on to higher orders the coefficients K_{rs} are no longer constants but again they are functions of $p(x)$ and $A(x)$ only. Hence it is possible to transfer information from the upstream region into the wake downstream and to formulate a new outer boundary condition for the boundary-value problem (7), (9) independently of the nonlinear interaction process which takes place in the region $x, y = O(1)$. (Special applications of (14) can be found for example in Smith 1977, eq. (2.5*b*), with one minor misprint: D_4 should be replaced by $[\frac{1}{2}D_1 A(x)+D_4]$; Smith & Merkin 1982, eq. (2.2*d*)).

In the case of supersonic flow past profiles consisting of a flat plate far upstream, (3), the similarity solution (11) yields only the trivial solution $\bar{f}(\bar{\eta})\equiv 0$ owing to the vanishing pressure distribution $p(x\rightarrow-\infty)\equiv O$. The asymptotic structure of the flow for $x\rightarrow-\infty$ is given by the linearized form of the free-interaction solution first obtained by Lighthill (1953). Since this solution shows an exponential decay of all field quantities for $y\rightarrow\infty$, $\bar{K}_2 = 0$ in (13) and as a consequence expansion (14) includes no algebraic terms in the whole domain of the flow $(-\infty<x<\infty)$:

$$\Psi(x, y\rightarrow\infty) \sim \tfrac{1}{2}[y+A(x)]^2+p(x)+o(y^{-r}), \quad \forall r. \tag{15}$$

Comparison with (10), valid downstream, shows that the second complementary function must also vanish ($K_2 = 0$) thus leading to the new boundary condition for $f(\eta\rightarrow\infty)$, replacing (9*b*):

$$K_2 = \lim_{\eta\rightarrow\infty}\left(\frac{f-\eta f'-D}{1-3\beta}\,\eta^{-3\beta}\right) = \bar{K}_2 = 0. \tag{16}$$

With this condition the similarity solution (5), satisfying the boundary-value problem (7), (9*a*), (16) yields unique results for all values of the exponent β. (The only exceptions are body contours producing $p(x)\equiv 0$ not only for $x\rightarrow-\infty$ but also for $x\rightarrow\infty$; e.g. a step $F(x) = \alpha H(x)$, a hump $F(x)\sim\alpha\delta(x)$. Here $H(x)$ and $\delta(x)$ denote the Heavisides unit function and Dirac delta function, respectively.) These unique results – valid for disturbances $\alpha = O(1)$ – are in agreement with the asymptotic behaviour $(x\rightarrow\infty)$ of the full solution of the linearized lower-deck equations for $\alpha \ll 1$. This linear solution was first obtained for the corner-flow problem by Stewartson (1970, 1971) using Fourier transform theory. Using the same method for supersonic flow past general profiles (3) (see Gittler 1985) we get the wall-shear-stress distribution (cf. (5*e*))

$$\tau_w(x\rightarrow\infty) \sim 1+\alpha K_\tau x^{\beta-\frac{2}{3}}, \tag{17a}$$

where

$$K_\tau = -\frac{\Gamma(\frac{1}{3})\,\Gamma(\beta+2)}{3^{\frac{1}{3}}\Gamma(\frac{2}{3})\,\Gamma(\beta+\frac{1}{3})}.$$

The displacement thickness (cf. (6)) is given by

$$A(x\rightarrow\infty) \sim \alpha K_1 x^{\beta-\frac{1}{3}}, \tag{17b}$$

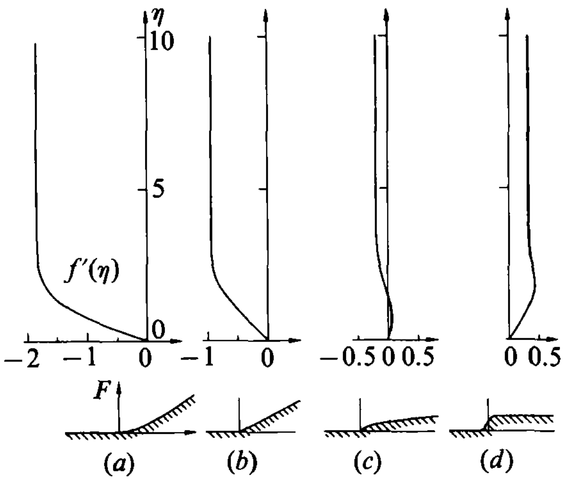

FIGURE 1. Disturbance of the u-component (equation (5c)) downstream on a profile (3) for different values of β: (a) $\beta = \frac{1}{2}$, (b) 0, (c) $-\frac{1}{2}$, (d) -1.

with
$$K_1 = -\frac{\Gamma(\frac{1}{3})\,\Gamma(\beta+2)}{3^{\frac{2}{3}}\Gamma(\beta+\frac{1}{3})}.$$

These results are valid for $\beta \neq -1-n$, $n \in N$, only. (For example, the special case $\beta = -2$, corresponding to a body contour characterized by $F(x \rightarrow \infty) \sim \alpha x^{-1}$, leads to the asymptotic relationship for the wall-shear stress distribution

$$\tau_w(x \rightarrow \infty) \sim 1 - \alpha \frac{10}{3^{\frac{7}{3}}\Gamma(\frac{2}{3})}[\ln x + \psi(-\tfrac{5}{3})]\,x^{-\frac{8}{3}}, \tag{18}$$

where $\psi(z) = \mathrm{d}/\mathrm{d}z \ln \Gamma(z)$ denotes the digamma function. This example showing the appearance of a ln-term explains the singularities in the solution (17) for $\beta = -1-1n$, $n \in N$).

As mentioned above, these asymptotic results due to linear theory ($\alpha \ll 1$) are in complete agreement with the similarity solution for $\alpha = O(1)$, equation (5). This was checked by solving the boundary-value problem (7), (9a), (16) analytically, yielding identical results: $f''(0) = K_\tau$ and $f'(\infty) = K_1$, see Gittler (1985). Some of these results for the disturbances of the u-component $f'(\eta)$ (equation (5c)) are displayed in figure 1. For the step-profile (figure 1d) the velocity distribution $f'(\eta)$ is a solution of the differential equation (7) with $\beta D = 0$ and $\beta = -1$, together with conditions (9a) and a specified value $f''(0)$ as a third inhomogeneous initial condition taken from the linear solution. If $\alpha > 0$ the similarity profiles show a retardation of the whole boundary-layer flow for $\beta = \frac{1}{2}$ and $\beta = 0$ (ramp), whereas supervelocities appear in the vicinity of the wall if $\beta < -\frac{1}{3}$ owing to the favourable pressure gradient (cf. (17a)). Downstream of a step we obtain positive velocity disturbances throughout the whole boundary layer with a pronounced maximum near the wall.

For supersonic motions along non-flat wall shapes ($F(x \rightarrow -\infty) \neq 0$) and in all subsonic flows the same condition $K_2 = \bar{K}_2$ (cf. the left-hand side of (16)) is sufficient to determine a unique similarity solution valid downstream. In all these cases the coefficient $\bar{K}_2$ of the algebraic term in the expansion (13) (provided by the upstream solution, (11), (12)), is non-zero and corresponds to a term $O(y^3\beta)$ which appears in the wall layer of the boundary-layer flow approaching the interaction region (cf. Brown & Stewartson 1970) and which persists in the main deck of the triple-deck structure ($Y \to 0$) for all x.

To decide, whether these unique results are indeed the leading term of the asymptotic structure for $x \to \infty$, it is necessary to consider the possible occurrence of homogeneous eigensolutions. For that purpose we extend (5):

$$\Psi(x \to \infty, y) \sim \tfrac{1}{2}y^2 + \alpha x^{\beta} f(\eta) + C x^{\lambda} h(\eta) + \ldots, \quad \eta = y/x^{\frac{1}{3}}. \tag{19}$$

The function $h(\eta)$ satisfies the homogeneous differential equation (cf. (7), (9))

$$h''' + \tfrac{1}{3}\eta^2 h'' - \lambda(\eta h' - h) = 0, \tag{20a}$$

subject to the initial conditions

$$h(0) = h'(0) = 0. \tag{20b}$$

If

$$h''(0) = 1 \tag{20c}$$

is chosen as a third initial condition, the solutions of the initial-value problem (20) exhibit the asymptotic behaviour (cf. (10))

$$h(\eta \to \infty) \sim K_1 \eta + K_2 \eta^{3\lambda} + K_3 \eta^{-3\lambda-4} \mathrm{e}^{-\eta^3/9} + \ldots, \tag{21}$$

with the value of the coefficient K_2 being a function of λ only: $K_2 = K_2(\lambda)$.

Solving (20) by means of hypergeometric functions and using their asymptotic properties (see Abramowitz & Stegun 1970) yields

$$K_2 = \frac{\Gamma(\frac{2}{3})\, 3^{-2\lambda+\frac{1}{3}}}{(3\lambda-1)\,\Gamma(\lambda+1)}. \tag{22}$$

Since the occurrence of any eigensolution is caused by the interaction process in the region $x = O(1)$ without any accompanying forcing pressure gradient, they must not include algebraic terms for $y \to \infty$, i.e. $K_2 = 0$. Therefore, the zeros of (22) determine the spectrum of the eigenvalues λ_k, which corresponds to the negative integers:

$$\lambda_k = -k, \quad k \in N. \tag{23}$$

The eigenfunctions $h_k(\eta)$ have to satisfy the orthogonality condition

$$\frac{1}{R_k}\int_0^{\infty} \eta^4 \mathrm{e}^{\eta^3/9} \left(\frac{h_k}{\eta}\right)' \left(\frac{h_l}{\eta}\right)' \mathrm{d}\eta = \delta_{kl}, \tag{24}$$

where δ_{kl} denotes the Kronecker delta and R_k is an appropriate real number. The derivatives of the first ten eigenfunctions $h_k'(\eta)$, corresponding to the velocity disturbances $u \sim C_k x^{\lambda_k - \frac{1}{3}} h_k'(\eta)$, are depicted in figure 2. Finally it should be mentioned that no eigensolutions governed by the homogeneous form of (12) with $\bar{D} = 0$ can appear in the upstream region for $x \to -\infty$. This is quite clear for physical reasons and can be proved analytically.

In supersonic flow the contribution of the first eigenfunction

$$C_1 x^{-1} h_1(\eta), \quad \text{with} \quad h_1(\eta) = \frac{\eta}{2}\int_0^{\eta} \mathrm{e}^{-s^3/9}\,\mathrm{d}s, \tag{25}$$

forms the leading term of the expansion (19) downstream of a profile (3) with $\beta \leqslant -1$. Apart from the unknown coefficient C_1 this first eigensolution is identical with the linear solution downstream of a step in supersonic flow (figure 1*d*). Since the

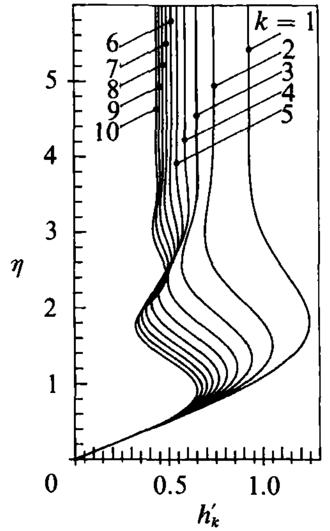

FIGURE 2. Derivatives $h'_k(\eta)$ of the first ten homogeneous eigenfunctions, corresponding to the velocity disturbances $u(x\to\infty, y) \sim C_k x^{-k-\frac{1}{3}} h'_k(\eta)$, with $\eta = y/x^{\frac{1}{3}}$.

coefficient C_1 cannot be predicted in advance, we are now finally able to define the interval for the exponent β in which a similarity solution (19) can be specified uniquely. For two-dimensional supersonic boundary-layer flow we obtain:

$$-1 < \beta \leqslant \tfrac{2}{3}, \quad M_\infty > 1. \tag{26a}$$

Therefore we have a uniquely defined asymptotic behaviour downstream along the profiles (*a*), (*b*) and (*c*) of figure 1, whereas the amplitude of the velocity disturbance for profile (*d*) remains indefinite.

In the case of subsonic motion along a body contour (3) with $\beta = -1$ (step profile: $F(x) = \alpha H(x)$) the expansion takes the similarity form $\psi \sim \frac{1}{2}y^2 + \alpha x^{-1} \ln x f(\eta)$ due to the confluence of the forcing pressure term and the first eigensolution. (The same effect caused the appearance of the logarithmic term in (18)). No subsonic boundary-layer flow is possible for $\beta = \frac{2}{3}$ since the similarity solution (11) holding upstream contradicts the initial condition of the lower-deck equation (1*d*) in this special case. So the range of unique solutions is given by

$$-1 \leqslant \beta < \tfrac{2}{3} \quad \text{for} \quad M_\infty < 1. \tag{26b}$$

Other pressure-displacement relationships (cf. (1*f*), (1*g*)) lead to results which differ from the estimates (26*a*) and (26*b*). For example in axisymmetric, supersonic flow along profiles (3) (cf. Kluwick, Gittler & Bodonyi 1984, 1985) we obtain $0 \leqslant \beta \leqslant \frac{5}{3}$).

Finally we should mention again that for values of β yielding unique solutions (26*a*, *b*) the asymptotic behaviour is identical with the results of linear theory in the limit $x \to \infty$. ($M_\infty > 1$: equation (17); $M_\infty < 1$: e.g. corner flow problem ($\beta = 0$), see Stewartson 1970, 1971).

Before discussing the applications of the results given above it should be mentioned that Libby & Fox (1963) calculated eigenfunctions occurring as perturbation solutions of the whole Blasius boundary layer. The velocity disturbances specified by these eigensolutions (Libby & Fox 1963, figure 1*a*) obviously have to vanish at the outer edge of the laminar boundary layer, but nevertheless they show a strong resemblance to the results depicted in figure 2. Owing to the scaled variable $\eta = y/x^{\frac{1}{3}}$ the triple-deck eigenfunctions spread out (away from the wall) for $x \to \infty$ and so the transition into the Blasius eigenfunctions downstream of the interaction process is evident.

4. Numerical confirmation and application of the results

Let us consider the elementary, but nevertheless important, interaction process caused by a hump $F(x)$ on an otherwise flat surface in supersonic flow:

$$F = \alpha(1-x^2)^2, \quad |x| \leqslant 1, \tag{27a}$$

$$F \equiv 0, \quad |x| > 1. \tag{27b}$$

Since we have no forcing pressure gradient, the leading terms of the expansion for the stream function contain homogeneous eigensolutions only:

$$\Psi(x \to \infty, y) \sim \tfrac{1}{2}y^2 + C_1 x^{-1} h_1(\eta) + C_2 x^{-2} h_2(\eta) + \dots, \quad \eta = y/x^{\frac{1}{3}}, \tag{28}$$

with $h_1(\eta)$ given in (25) and $h_2(\eta) = \frac{1}{2}\eta \int_0^\eta (1 - s^3/15)\, e^{-s^3/9}\, ds$.

Nonlinear terms – starting with $C_1^2 x^{-\frac{8}{3}} f_{\frac{8}{3}}(\eta)$, where $f_{\frac{8}{3}}(\eta)$ is governed by an inhomogeneous boundary-layer problem – and other higher-order terms are omitted here. For small disturbances, $\alpha \ll 1$, linear theory (e.g. Smith 1973) yields

$$C_1 = 0, \quad C_2 = -\frac{32}{3^{10/3}\Gamma(\frac{2}{3})}\alpha, \tag{29}$$

hence it follows that $\tau_w - 1 = O(x^{-\frac{8}{3}})$ and $A = O(x^{-\frac{7}{3}})$ for $x \to \infty$, cf. (5) and (6).

To show the appearance of the first eigenfunction ($C_1 \neq 0$) in the wake downstream of a nonlinear interaction process ($\alpha = O(1)$) the lower-deck equations had to be solved numerically. The system of equations and boundary conditions (1*a–f*), (27) was cast into finite-difference form using centred differences in the normal direction and the implicit Crank–Nicolson scheme of second order for the streamwise direction. An automatically working shooting algorithm (see Daniels 1974; Gittler & Kluwick 1987) yields a solution in which the occurrence of an expansive or compressive eigensolution is pushed downstream as far as possible. This shooting method – working in supersonic flow only – needs no information about the behaviour of the field quantities for $x \to \infty$, thus yielding accurate results far downstream which can be used to check the validity of the asymptotic expansion (28). To establish the asymptotic structure in the numerical results it was necessary to increase the accuracy of the computations considerably. At every x-station the Newton iterations had to be repeated until the corrections of all field quantities were less than 10^{-12} (instead of the usual value of 10^{-6}). Typical step sizes are $\Delta x = 0.05$ and $\Delta y = 0.2$. In order to pursue the calculations further downstream it was found necessary to restart the shooting algorithm at a suitably chosen x-station by interpolation between the two diverging compressive and expansive solutions (for example, near $x = 14$ in figure 3).

The displacement thickness $A(x)$ turned out to be the most useful field quantity for calculating the coefficients C_1 and C_2 in (28). The asymptotic expansion of $A(x)$ according to (28) is given by

$$A(x \to \infty) \sim A_1(1 + a_1 x^{-\frac{4}{3}})\, x^{-\frac{4}{3}} + A_2(1 + a_2 x^{-\frac{4}{3}})\, x^{-\frac{7}{3}} + \dots, \tag{30}$$

where
$$A_1 = \frac{\Gamma(\frac{1}{3})}{3^{\frac{1}{3}}2} C_1, \quad A_2 = \frac{2\Gamma(\frac{1}{3})}{3^{\frac{1}{3}}5} C_2$$

and $a_1 = -10\Gamma(\frac{2}{3})/3^{\frac{5}{3}}$, $a_2 = 2a_1$.

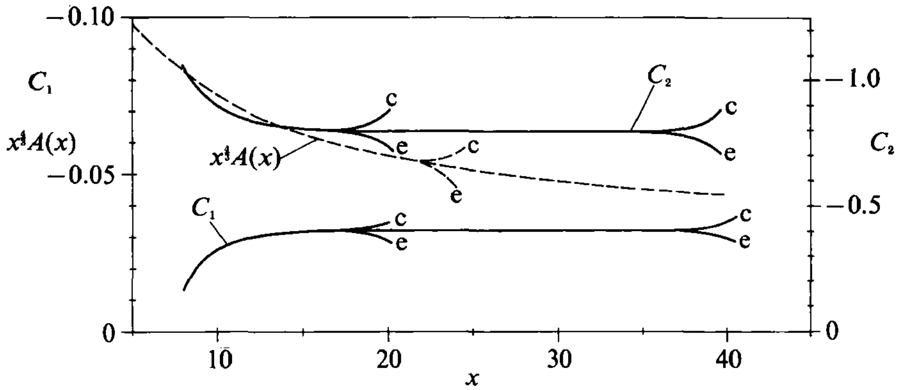

FIGURE 3. Displacement thickness $A(x)$ and coefficients C_1 and C_2, according to (28) and (30), downstream of a nonlinear interaction process caused by a hump (27) with $\alpha = 1$, in supersonic flow; c denotes compressive free interaction; e denotes expansive free interaction.

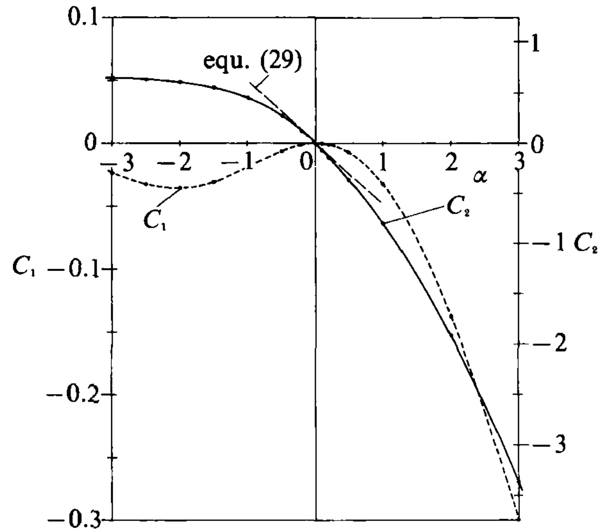

FIGURE 4. Coefficients C_1 and C_2 characterizing the asymptotic structure of the wake of an interaction process along the profile (27) for different values of α; also shown is the linear solution for C_2, equation (29).

The pressure distributions evoked by the eigenfunctions h_1 and h_2 in (28) give rise to the additional terms containing the coefficients a_1 and a_2. However, since these disturbances do not cause contributions to the displacement thickness, the process terminates and no further terms appear inside the parentheses of (30) in the case of supersonic flow.

In connection with (30) and its first derivative, the numerical results for $A(x)$ and $A'(x)$ can be used to determine the coefficients C_1 and C_2, which are depicted in figure 3 for the case $\alpha = 1$. As can be seen in this figure, C_1 and C_2, both approach constant values, which confirms the asymptotic structure (28) holding downstream. These computations were repeated for different values of α and the results are shown in figure 4. For $|\alpha| \ll 1$ we observe the linear variation of C_2 given in (29) but for all values of $\alpha \neq 0$ we obtain a non-zero coefficient C_1. Therefore, in contrast to linear theory, the asymptotic behaviour of the field quantities far downstream is always dominated by the first, slowest decaying eigensolution whose amplitude C_1 cannot be predicted in advance without having obtained a full numerical solution of the problem.

It is interesting to note that the structure of the wake is characterized by $C_1 < 0$, as can be seen in figure 4. Therefore, an observer located far downstream cannot decide whether the boundary layer has been disturbed by a hump ($\alpha > 0$) or an indentation ($\alpha < 0$), at least in the interval $-3 < \alpha < +1$.

Before closing this section let us briefly outline some possible applications of the results discussed so far.

(i) The nonlinear interacting boundary-layer flow over a ramp ($\beta = 0$, corner flow problem) was studied first by Rizzetta, Burggraf & Jenson (1978), and Smith & Khorrami (1991) and Smith & Merkin (1982) for supersonic and subsonic Mach numbers, respectively. Using the asymptotic behaviour given by the uniquely determined similarity solutions in this case it seems possible to improve the accuracy of the numerical computations for these problems.

(ii) A number of recent papers describe the interaction of free-stream disturbances and boundary-layer flow past an obstacle. In all these 'receptivity problems' it is necessary to have a very accurate solution of the steady nonlinear triple-deck flow whose stability is investigated. An important point concerning the accuracy of finite-difference computations is the handling of the interaction law between the pressure and the displacement in incompressible flow (1*g*). Also, in subsonic motions along profiles (3) with $\beta < -1$, see (26*b*), the first eigensolution provides the leading term downstream. Therefore in all subsonic interactions evoked by a surface-mounted obstacle the far-field tails of the Hilbert integral (1*g*) should be evaluated using the asymptotic relationship $A(x \rightarrow \infty) = O(x^{-\frac{4}{3}})$ instead of the wrong behaviour $A = O(x^{-\frac{7}{3}})$ given by linear theory (see Smith & Bodonyi 1985 and Bodonyi *et al.* 1989).

(iii) Another application concerns the modification of a spectral method for the solution of the lower-deck equations. Spectral methods have the important advantage of treating reversed-flow regions correctly, without the need for any kind of approximation or cumbersome adaptation required in conventional schemes. The original version of this method introduced by Burggraf & Duck (1981) can be applied only if the transformed variable

$$\bar{\tau}(\omega, y) = \int_{-\infty}^{\infty} [\tau(x, y) - 1]\, \mathrm{e}^{-\mathrm{i}\omega x}\, \mathrm{d}x \rightarrow 0 \quad \text{for} \quad \omega \rightarrow 0 \tag{31a}$$

sufficiently fast so that the contribution for the discretized inverse Fourier integral in the interval around $\omega = 0$ vanishes,

$$\int_{-\Delta\omega/2}^{\Delta\omega/2} \bar{\tau}(\omega, y)\, \mathrm{e}^{\mathrm{i}\omega x}\, \mathrm{d}x = 0, \tag{31b}$$

and the point $\omega = 0$ must be excluded from the computational domain.

Since the contribution of the first eigenfunction yields $\bar{\tau}(\omega \rightarrow 0) = O((\mathrm{i}\omega)^{\frac{2}{3}})$ for every interaction process, we obtain a non-zero value for the integral (31*b*) which is taken into account in a modified scheme. As a consequence, the numerical results of this modified spectral method automatically exhibit the correct asymptotic properties far downstream (see Gittler 1984, 1985). This is shown in figure 5 for the hump profile (27) with $\alpha = 3$ in supersonic flow. The wall shear stress τ_w obtained by means of the modified spectral method is in very good agreement with the results of the finite-

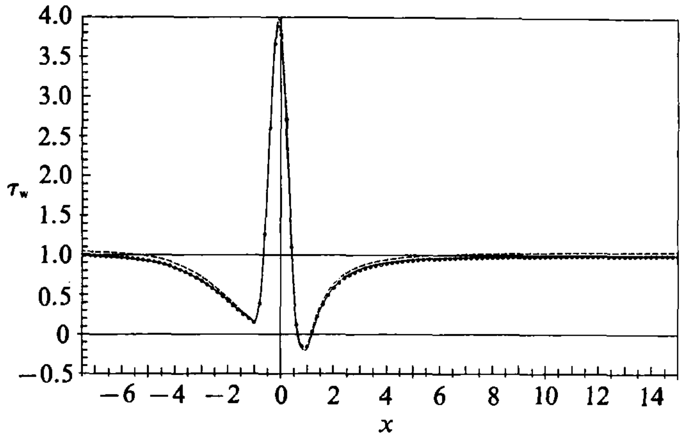

FIGURE 5. Wall-shear-stress distribution $\tau_w(x)$ for the hump profile (27) with $\alpha = 3$, in supersonic flow: ------, spectral method, Burggraf & Duck (1981); oooooo, modified spectral method, Gittler (1984, 1985); ———, finite-difference method.

difference method, mentioned above, whereas the original spectral scheme yields positive τ_w-disturbances in the upstream interaction region as well as in the wake downstream, which are both physically incorrect. In contrast to the original version, the modified scheme is also able to cope with boundary-layer interactions induced near corners where the transformed quantity $\bar{\tau}$ exhibits a singularity $\bar{\tau}(\omega \to 0) = O((\mathrm{i}\omega)^{-\frac{1}{3}})$, see Gittler & Kluwick (1989) for further details.

5. Nonlinear similarity solution

As already mentioned in §2 the value $\beta = \frac{2}{3}$, corresponding to the body contour $F \sim \alpha x^{\frac{5}{3}}$, represents an upper limit for the exponent β. In this case the two-term expansion of the stream function (5a) is valid for $\alpha \ll 1$ only, since the shear-stress disturbances (5e) do not decay for $x \to \infty$ but remain constant. As might be expected, it is possible to construct a nonlinear similarity solution of the interaction process in this special case.

The appropriate asymptotic form for the stream function is

$$\Psi(x \to \infty, y) \sim x^{\frac{2}{3}} g(\eta), \quad \eta = y/x^{\frac{1}{3}}. \tag{32a}$$

With the field quantities

$$u \sim x^{\frac{1}{3}} g'(\eta), \quad v \sim x^{-\frac{1}{3}}[-\tfrac{2}{3}g(\eta) + \tfrac{1}{3}\eta g'(\eta)], \quad \tau \sim g''(\eta) \tag{32b–d}$$

and the prescribed pressure

$$p \sim \alpha \tfrac{5}{3} x^{\frac{2}{3}} \tag{33}$$

the momentum equation (1b) and the conditions (1c) and (1e) take the form

$$g''' + \tfrac{2}{3}gg'' - \tfrac{1}{3}g'^2 - \tfrac{10}{9}\alpha = 0, \tag{34a}$$

$$g(0) = g'(0) = 0, \tag{34b}$$

$$g''(\eta) \to 1 \quad \text{as} \quad \eta \to \infty. \tag{34c}$$

The nonlinear boundary-value problem (34) is of the Falkner–Skan type with a novel boundary condition (34c) corresponding to the linear velocity distribution at the outer edge of the lower deck, (1e).

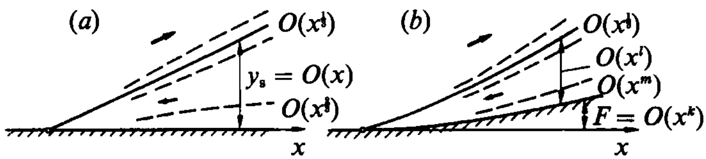

FIGURE 6. Flow structure of a self-induced supersonic separation on (*a*) a flat plate and (*b*) a profile $F(x \to \infty) \sim \alpha x^k$.

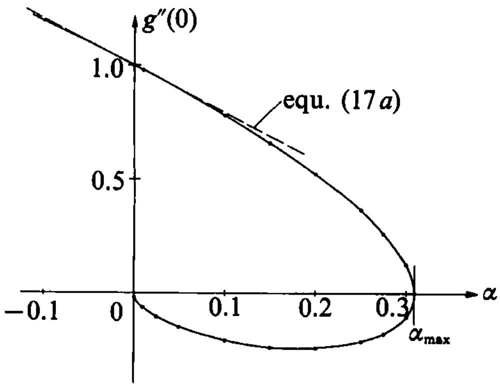

FIGURE 7. Solution of the boundary-value problem (34): wall shear stress $\tau_w = g''(0)$ versus α. -----, Linear theory (equation (17*a*)) with $\beta = \frac{2}{3}$).

Before discussing the numerical solution of (34) let us draw attention to another argument leading to the same Ansatz (32*a*). Neiland (1969) and Stewartson & Williams (1973) gave an asymptotic description of the self-induced separation of a supersonic boundary layer. The multistructured lower deck, shown in figure 6(*a*), consists of a shear layer of thickness $O(x^{\frac{1}{3}})$ which is centred around the straight separation line $y_s = P_0 x$, a slow inviscid reversed flow below y_s, and a sub-boundary layer of thickness $O(x^{\frac{2}{3}})$, which enables the no-slip condition to be satisfied on the plate. For the generalized problem of supersonic separated flow along a profile $F \sim \alpha x^k$ a similar description of the flow field is possible, figure 6(*b*). The thickness of the separation region $O(x^l)$ and of the wall boundary layer $O(x^m)$, which are determined by matching arguments, both decrease with increasing values of $k > 0$. An interesting detail of these solutions is the algebraically growing velocity at the outer edge of the backward-moving wall layer, a behaviour which was studied first by Gajjar & Smith (1983) in the context of hydraulic jumps and hypersonic separation. In the limit $k \to \frac{5}{3}$ we get $l, m \to \frac{1}{3}$, so the wall layer, the region of backflow and the free shear layer merge and the single expression (32*a*) for the asymptotic description of the whole flow-field becomes possible, even in the case of separated flow.

Figure 7 shows the calculated wall shear stress $\tau_w = g''(0)$. Solutions of the boundary-value problem (34) exists for $\alpha \leqslant \alpha_{max} = 0.30905$ only. The occurrence of solutions with oscillatory behaviour for $\alpha > \alpha_{max}$ – analogous to the results of Libby & Liu (1967) for the Falkner–Skan equation – seems to be impossible for the boundary-value problem (34). In the vicinity of the undisturbed state ($\alpha = 0, g = \frac{1}{2}\eta^2$, $g''(0) = 1$) the linear solution, (17*a*), $\tau_w = 1 - \alpha 10 \Gamma(\frac{1}{3})/3^{\frac{7}{3}}$, provides a good approximation to the nonlinear results. If $\alpha > 0$, e.g. for compressive flow, the solution consists of two branches corresponding to attached and separated flow, respectively. In the limiting case $\alpha = \alpha_{max}$ the shear stress at the wall vanishes identically. The

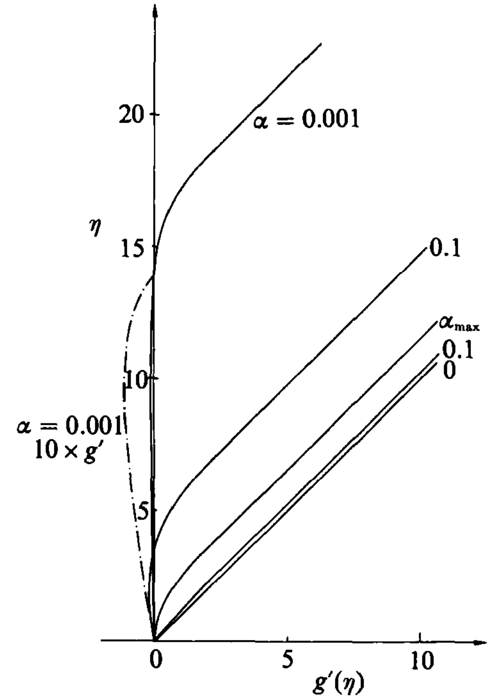

FIGURE 8. Velocity distribution $g'(\eta)$ for various values of α.

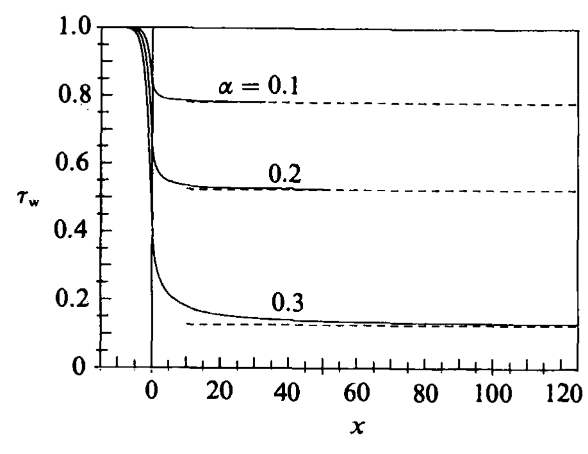

FIGURE 9. Wall shear stress distribution $\tau_w(x)$ along the profile $F(x) = \alpha x^{\frac{5}{3}}$ in supersonic flow for various values of α: ———, finite-difference method; ------, nonlinear similarity solution, (32).

velocity distributions $g'(\eta)$ for various values of α are depicted in figure 8. Results with positive wall shear ($g''(0) > 0, 0 < \alpha < \alpha_{max}$) exhibit small velocity disturbances near the wall only, while backflow solutions exhibit a rapidly increasing separation region for $\alpha \rightarrow 0+$.

Finite-difference solutions for supersonic flow along the profile $F(x) = \alpha x^{\frac{5}{3}} H(x)$ for $\alpha = 0.1$, 0.2 and 0.3 are displayed in figure 9. As can be seen, the wall shear stress reaches constant values downstream, in perfect agreement with the results of the nonlinear similarity solution. The numerical results for $\alpha = \alpha_{max}$ (not depicted in figure 9) also show a smooth wall shear stress distribution which is always positive with $\tau_w \to 0$ for $x \to \infty$. For $\alpha > \alpha_{max}$ no solutions can be found and thus a global change of the flow pattern is expected to take place. Most likely – as indicated by numerical solutions – the separation point is pushed upstream a distance which is infinite in triple-deck scaling. Thus, one is led to the interesting conclusion that a wall of the form

$$\frac{F^*}{L^*} = \begin{cases} \alpha_{max}\lambda^{\frac{4}{3}}(M_\infty^2 - 1)^{\frac{1}{2}}(T_\infty^*/T_w^*)(x^*/L^*)^{\frac{5}{3}} & \text{for} \quad x^* > 0 \\ 0 & \text{for} \quad x^* \leqslant 0, \end{cases} \tag{35}$$

with $\alpha_{max} = 0.30905$ generates the largest possible deflection of a laminar boundary layer in supersonic flow if separation is to be avoided. Here the superscript * characterizes dimensional quantities. L^* denotes the distance from the leading edge, M_∞ the free-stream Mach number, T_∞^* and T_w^* are the free stream and wall temperature, respectively, while λ is associated with the non-dimensional skin friction in the unperturbed laminar boundary layer (for the Blasius boundary layer on a flat plate, $\lambda = 0.3321$).

The corresponding pressure distribution – producing the sharpest pressure rise in the shortest distance – is given by

$$\frac{p^* - p_\infty^*}{\rho_\infty^* u_\infty^{*2}} = \begin{cases} \frac{5}{3}\alpha_{max}\lambda^{\frac{4}{3}}(T_\infty^*/T_w^*)(x^*/L^*)^{\frac{2}{3}} & \text{for} \quad x^* > 0 \\ 0 & \text{for} \quad x^* \leqslant 0, \end{cases} \tag{36}$$

where ρ_∞^* and u_∞^* denote the undisturbed values of the density and velocity, respectively. The result (36) was derived in the case of incompressible flow by Stratford (1954) based on more heuristic arguments in a paper concerning laminar-boundary-layer flow near separation. A body contour which produces the pressure distribution (36) inside the interaction region in subsonic flow is given by the convex profile

$$\frac{F^*}{L^*} = \begin{cases} -(2/\sqrt{3})\,\alpha_{max}\lambda^{\frac{4}{3}}(1-M_\infty^2)^{\frac{1}{2}}(T_\infty^*/T_w^*)(-x^*/L^*)^{\frac{5}{3}} & \text{for } x^* \leqslant 0 \\ -(1/\sqrt{3})\,\alpha_{max}\lambda^{\frac{4}{3}}(1-M_\infty^2)^{\frac{1}{2}}(T_\infty^*/T_w^*)(x^*/L^*)^{\frac{5}{3}} & \text{for } x^* > 0, \end{cases} \tag{37}$$

as can be seen from (2), (3) and (4). Owing to the relation $F^*(x^*) = \frac{1}{2}F^*(-x^*)$ for $x^* > 0$ the pressure distribution upstream vanishes identically, making possible the existence of a triple-deck flow in this limiting case for $\alpha < \alpha_{max}$, cf. the remark preceding (26*b*).

Finally we would like to show the significance of the present results for the recently raised question of a termination of the triple-deck structure for increasing disturbances of the flow. For example, the results of Smith & Khorrami (1991) show a breakdown of the triple-deck solution for the supersonic ramp flow problem if the scaled ramp angle α approaches a critical value α_s. This local breakdown is caused by the formation of a singularity in the reversed-flow region. From the findings of this section it is clear, however, that separation can be avoided for arbitrary ramp angles α if the profile $F(x) = 0.30905(x-x_1)^{\frac{5}{3}}$ is used to smooth the concave sharp corner in the interval between $x_1 = -1.082\alpha^{\frac{3}{2}}$ and $x_2 = 1.623\alpha^{\frac{3}{2}}$. This yields a wall-shear-stress distribution which is positive along the whole contour. This method can be applied to all profiles with $\beta < \frac{2}{3}$. Therefore, in all these cases the local reversed-flow breakdown is removable and we have no finite upper limit for the controlling parameter α. This is also in complete agreement with the fact that there always exists a unique asymptote for all values of α.

In contrast, a non-local breakdown of the two-dimensional, steady triple-deck flow is evoked by the profile (35) (resp. (37) for subsonic flow), only in this case ($\beta = \frac{2}{3}$) the cutoff value α_{max} is determined by both the asymptotic solution valid in the far field and the full numerical solution of the triple-deck equations. Therefore, the breakdown of the interacting laminar-boundary-layer approach for $\alpha \geqslant \alpha_{max}$, $\beta = \frac{2}{3}$ or $\beta > \frac{2}{3}$, respectively, is of a global form which cannot be removed or avoided.

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